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Collective phenomena in large populations of globally coupled relaxation oscillators

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We present a study of a large pool of globally coupled relaxation oscillators. The reaction of the pool to the presence of a modulating external field is discussed. The coupling is assumed homogeneous and linear. Randomly distributed internal frequencies introduce a disordering element that, due to the coupling, can result in oscillator quiescence. Self-synchronization is shown to be absent in this system. However, this is entirely due to the linear coupling. For identical oscillators the basic state is incoherent and marginally stable in an extended region of parameter space. With modulation on the levels, the average rotation number as function of the external frequency lies on a devil's staircase, as for a single oscillator. However, the locked regions shrink with increasing coupling. This has some important consequences for the critical lines of the averaged system. In the case of modulation on the frequency (no damping), the average rotation number is still independent of the external signal, as for a single oscillator. The real surprise lies in the resulting distribution of individual frequencies or rotation numbers. No matter how the external field is applied, this distribution is forced into a devil's staircase exhibiting a critical point, which in the case of modulation on the lower level is a transition from quasiperiodicity to chaos and otherwise to completeness.

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I. INTRODUCTION

Recently collective phenomena in large pools of coupled oscillators have attracted much attention [1–4]. One of the interesting features of such populations is the possibility of spontaneous synchronization, which in nature has been observed in a wide range of physical, chemical, biological, and medical systems. Examples are charge-density waves [5], oscillating chemical reactions [6], fireflies flashing in unison [7,8], the human circadian rhythm [9], and an audience applauding the prima ballerina. The analysis of these large systems with many degrees of freedom has involved elements from both statistical mechanics and nonlinear dynamics.

Self-synchronization of pools of oscillators has been a subject for a great deal of research for quite a while. The breakthrough was obtained by Winfree [10] who assumed that the interaction between the pool members was weak compared to the attraction of the individual oscillators to their limit cycles, so that amplitude variations could be neglected and only phase variations need to be considered (the phase rotator model). The important result here was that self-synchronization was a cooperative phenomenon, analogous to a phase transition. However, this and most of the following work relates to limit-cycle oscillators described by ordinary differential equations. Recently, a simple model for synchronous firing of relaxation oscillators has been investigated [11]. The model consists of a pool of identical "integrate and fire" oscillators globally coupled by the pulses they emit when they fire. When one oscillator fires, the pulse pulls all the other oscillators towards their firing level and if this is surpassed, they, too, will fire. The main result given in this investigation was that a state is obtained where the oscillators fire in synchronism. The time for obtaining synchrony has been studied in a recent experiment on 15 pulse-coupled electronic relaxation oscillators with a triangular time development [12].

We report a study of coupled relaxation oscillators with random internal frequencies in the presence of an alternating external field. For convenience the coupling is chosen to be of infinite range and homogeneous. This choice of global coupling may seem unrealistic but is justified in its relative simplicity. It is often found that an infinite-range model captures the essentials of more refined models. However, a few physical systems described well by infinite-range couplings do exist. As an example, we mention an array of resistively shunted Josephson junctions [13,14]. In fact, arrays of resistively coupled high-current-density superconducting microbridges in the flux flow regime (especially high-T_c bridges) and exposed to a microwave field can be modeled by relaxation oscillators with a linear global coupling and modulation on the current (corresponding to modulation on the frequency in the present notation).

The present work differs from previous studies by not only considering a pool of relaxation oscillators as in [15] but also using an ever-present linear global coupling term instead of pulse coupling. This kind of coupling may be of importance for coupled electronic oscillators and for earthquakes and swarm earthquakes. Also for many biological systems this kind of coupling may be present. One conclusion we draw is that this type of interaction does not allow for self-synchronization, although this is entirely due to the linear coupling. A highly interesting observation is the existence of stationary states that are marginally stable in extended regions of phase space. Furthermore, when the oscillators are nonidentical the interaction may result in oscillator quiescence. When an external modulating field is included synchronization to the field takes place in a way that always involves synchronization of some subset of the individual oscillators but not necessarily of the pool as a whole. Actually the average behavior of the pool may be completely unaffected by the presence of an external field. The predictions of the paper are at the moment being investigated in an experiment on a pool of between 16 and 64 resistively coupled electronic relaxation oscillators with instant reset and modulation on either levels or frequency.

The paper is organized as follows. In Sec. II we set forth the basic elements of the model. In Sec. III we define some useful general quantities and lay the foundation for the later stability analysis. The properties of the systems without modulation are treated in Sec. IV (identical oscillators) and Sec. V (nonidentical oscillators). In Sec. IV the analysis is extended to a repulsive interaction and the effect of damping is considered. Section V is mainly devoted to the question of oscillator quiescence. The effect of modulation is discussed in Sec. VI (identical) and Sec. VII (nonidentical). The main effort is to derive, if possible, critical lines for the existence of phase lock to the external field. Finally, we present our conclusions in Sec. VIII. Some general remarks about the problems involved in the implementation of numerical simulations are presented in Appendix A, while Appendix B deals with some preliminary results on the consequences of nonlinear coupling.

II. THE MODEL

The individual oscillators are characterized by the relaxation of a "voltagelike" state variable with instant reset to a lower level when a firing threshold is reached. The investigation is limited to the case of identical levels for all oscillators, while frequencies may vary. Denoting the state variable of the ith oscillator in the pool by x_i we have

$$\dot{x}_i=\omega_i-\Gamma x_i+A(t)+\frac{K}{N}\sum_{j=1}^{N}(x_j-x_i)\ , \quad i=1,\ldots,N\ , \tag{1}$$

with the firing condition

$$x_i(t^+)=T_{\rm bot} \quad \text{for } x_i(t)=T_{\rm top}\ . \tag{2}$$

The number of oscillators N is assumed to be very large, while the damping is assumed to be nonnegative. The quantity ω_i, which is always assumed to be positive, is the random intrinsic or natural frequency of the ith oscillator when no damping is present, i.e., $\Gamma=0$. Finally, K is the coupling strength which too is assumed to be nonnegative in most of the paper. The external field $A(t)$ has zero time average and is here assumed sinusoidal (amplitude a) although other choices could be of interest. The levels $T_{\rm top}$ and $T_{\rm bot}$ denote, respectively, the top and bottom thresholds which may be modulated by an external signal similar to $A(t)$. Without loss of generality, we can set the frequency of the external field to unity and also assume $\langle T_{\rm top}\rangle=1$ and $\langle T_{\rm bot}\rangle=0$, where $\langle\ \rangle$ denotes time average.

Defining the oscillator mean-field strength as $s=(1/N)\sum_{j=1}^{N}x_j$, the equations of motion can be written in the form

$$\dot{x}_i=\omega_i-(K+\Gamma)x_i+Ks(t)+A(t)\ . \tag{3}$$

With the above definition one can equally well think of the coupling as being between any two individual oscillators as between any oscillator and the mean field s. In general this mean-field approach is not possible. However, when the coupling is linear analytical progress is to some extent possible which is the foremost reason for choosing this kind of continuous coupling.

At first sight one might think that the attractive interaction will lead to self-synchronization. However, this does not take into account the effect of the abrupt firing events. Just after some oscillator fires it will still attract those close to firing but now with the effect of turning them away from following its own example. In essence it repels them from its own trajectory, creating momentarily a repulsive interaction. The outcome is thus a result of these two conflicting mechanisms.

III. GENERAL REMARKS

An important quantity in the following is the rotation number defined for a single oscillator as the average number of firings per unit time. Thus $R_i=1/T_i$, where T_i is the average time between firings for the ith oscillator. When no external field is present the rotation number for a single oscillator without damping becomes $R=1/T=\omega$, where T is the period. The average rotation number R for the pool is defined by

$$R\equiv\frac{1}{N}\sum_{i=1}^{N}R_i=\langle F(t)\rangle\ , \tag{4}$$

where $NF(t)dt$ is the number of firings by members of the pool in the time inverval $[t,t+dt]$.

For the sake of later stability calculations, let us consider two parallel oscillator trajectories displaced by an infinitesimal distance δ_1. Referring to Fig. 1 let us assume that the first trajectory hits the upper level at time

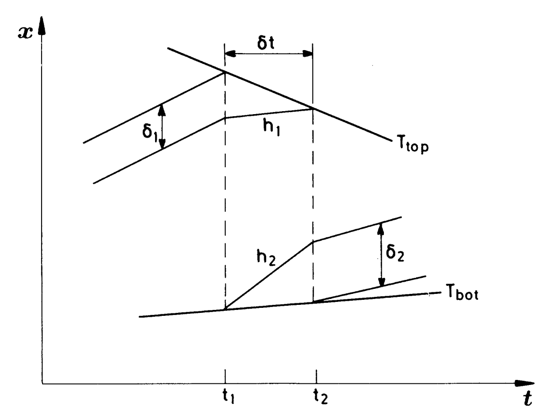

FIG. 1. Construction for calculating the relative change δ_2/δ_1 in distance between two oscillator trajectories during a firing event for use in stability calculations.

t_i and is reset to the lower level. From here it travels on with slope $h_2(t_i)$. The second trajectory travels on with slope $h_1(t_i)$ from time t_i to time $t_i+\delta t$ at which time it is reset to the lower level. Quite generally the distance δ_2 at time $t_i+\delta t$ is related to δ_1 through

$$\frac{\delta_2}{\delta_1}=\frac{h_2(t_i)-\dot{T}_{\mathrm{bot}}(t_i)}{h_1(t_i)-\dot{T}_{\mathrm{top}}(t_i)} . \tag{5}$$

The Lyapunov number is now defined as

$$\lambda=\prod_{i=0}^{n}\frac{\delta_2(t_i)}{\delta_1(t_i)}\Psi(t_{i+1}-t_i) , \tag{6}$$

where $\Psi(t_{i+1}-t_i)$ is the relative change in δ_2 until the next firing. Calculating the Lyapunov number thus involves knowledge of the firing times t_i and the appropriate slopes at the firing times. Evidently, calculation of the Lyapunov number involves contributions from both the continuous and the discontinuous (the firings) parts of the trajectory quantifying the arguments of Sec. II.

From Eq. (3) we can determine the time derivative of s by summing over i and dividing by the number of oscillators N. Defining the average frequency as $\varpi\equiv(1/N)\sum_{i=1}^{N}\omega_i$ we find

$$\dot{s}=\varpi-\Gamma s+a\sin(2\pi t)-[T_{\mathrm{top}}(t)-T_{\mathrm{bot}}(t)]F(t) , \tag{7}$$

where the last term results from the firing action. Since s is limited between T_{top} and T_{bot} taking the time average results in

$$\varpi-\Gamma\langle s\rangle-\langle[T_{\mathrm{top}}(t)-T_{\mathrm{bot}}(t)]F(t)\rangle=0 . \tag{8}$$

If the limiting levels are independent of time this equation simplifies into

$$\varpi-\Gamma\langle s\rangle-\langle F(t)\rangle=0 , \tag{9}$$

where we have used the normalization for the levels. Using the definition given in Eq. (4) for the average rotation number gives

$$R=\varpi-\Gamma\langle s\rangle . \tag{10}$$

Thus this simple result holds as shown for modulation with zero time average on the natural frequency (a nonzero time average will just produce an offset in ϖ). However, for modulation on the levels the argumentation evidently breaks down.

Without modulation an obvious guess is that the stable state for the pool of oscillators will be a stationary (incoherent) state with constant mean-field strength s. The effect of this assumption is to shift the natural oscillatory frequency and introduce an additional damping term. Quite obviously the state has a uniform distribution of firing times.

With the initial condition $x_i(t=0)=0$ the solution for x_i is

$$x_i=\frac{\omega_i+Ks}{K+\Gamma}\{1-\exp[-(K+\Gamma)t]\} , \tag{11}$$

with the period T_i determined by the condition $x_i(t=T_i)=1$:

$$T_i=-\frac{1}{K+\Gamma}\ln\left[1-\frac{K+\Gamma}{\omega_i+Ks}\right] . \tag{12}$$

To obtain running solutions we must require $\omega_i+Ks>K+\Gamma$. To proceed further s has to be calculated self-consistently.

With modulation, but all oscillators assumed identical, another obvious solution is the one where all the oscillators run in unison (are synchronized). In this case the coupling term is identical zero and the behavior of s is identical to that of a single oscillator which is known in great detail (see, e.g., [16]). One interesting question is whether self-synchronization can take place in the pool without an external field present.

IV. IDENTICAL OSCILLATORS, NO MODULATION

Since we cannot solve the system of equations analytically in general without preknowledge of the solution for s, we shall consent ourselves with discussing the above special guesses for solutions. Since the numerical simulations show the state having s constant to be the stable state we shall treat this case first in Sec. IV A. Thereafter, we discuss the self-synchronized state in Sec. IV B, before closing the section with a discussion of the effects of a finite amount of damping.

A. The incoherent stationary state

We shall now treat the case with $\omega_i=\omega$ and no damping, i.e., $\Gamma=0$. Furthermore, we assume s constant. The resulting state is thus a stationary state with a time-independent oscillator distribution with a uniform distribution of firing times t_j, i.e., the time development of any oscillator can be written in the form $x_j=f(t-t_j)$.

Because of this the calculation of s can simply be performed as the time integral of one oscillator over one period. Using the results for x_i and T_i given in Eqs. (11) and (12) with the above assumptions we find

$$\langle x_i\rangle=\frac{1}{T}\int_0^T x_i\,dt=\frac{\omega}{K}+s-\frac{1}{TK} , \tag{13}$$

which, using the definition of s, results in $T=\omega^{-1}$ in agreement with Eq. (10). Thus the frequency of the oscillator is unchanged by the coupling although the trajectory has changed. The mean field s can now be found from the equation for the period T:

$$s=\frac{1}{1-\exp(-K/\omega)}-\frac{\omega}{K} , \tag{14}$$

showing that s grows from $\frac{1}{2}$ at $K=0$ to 1 at $K=\infty$. Figure 2 shows the time development of one oscillator ($\omega=1$) for $K=3.5$, and compares it to that of an uncoupled free running oscillator. Also shown in this figure is the corresponding value of $s\simeq0.745$, while the fully drawn curve in Fig. 3 shows s as a function of K.

We shall discuss the stability of the above solution in some detail looking at a perturbation on a single oscillator whose trajectory has been displaced by an infinitesimal amount δ_1 while leaving s constant. Using the definitions of Fig. 1 with the trajectories denoting, re-

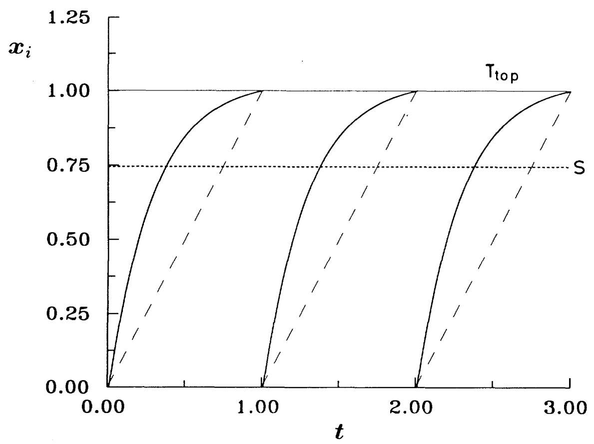

FIG. 2. The time development of a single oscillator with (full line $K=3.5$) and without (dashed line) coupling. The value of the mean-field strength s with coupling is also shown.

spectively, the state of the perturbed and the unperturbed oscillator, the ratio between the perturbations immediately before and after the firings is found from Eq. (5) by inserting the slopes as calculated from Eq. (3):

$$\frac{\delta_2}{\delta_1}=\frac{\omega+Ks}{\omega+Ks-K} . \tag{15}$$

Because of the damping introduced by the coupling, the perturbation will decrease in a period by the amount $\exp(-KT)$. Using the equation for T we find that the perturbation grows in one period with a factor $\lambda=\delta_2/\delta_1 \exp(-KT)=1$. The state is thus marginally stable.

To further investigate the possible stability of the state one would have to go to higher order in the stability analysis. However, numerical simulations show that the state after a perturbation that displaces a group of oscillators from a certain distribution of, e.g., firing times versus oscillator index will relax back into a distribution that after a permutation of indices is identical to the original distribution. The state is thus apparently stable but highly degenerate. A more careful study of this problem using a continuous-density approach (to be published elsewhere) reveals that the stationary state is indeed marginally stable.

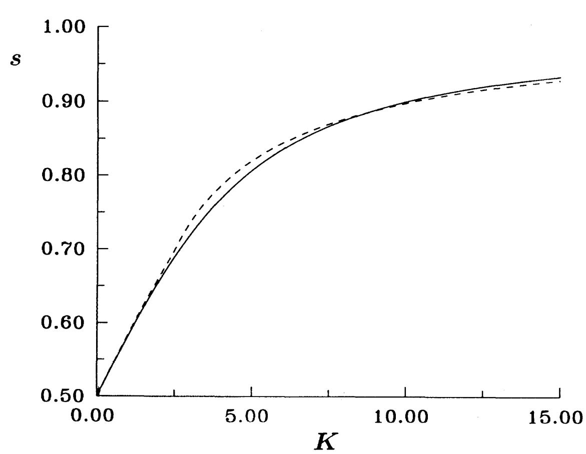

FIG. 3. The mean-field oscillator strength $s(K)$. Fully drawn line: identical oscillators ($\omega=1$). Dashed line: nonidentical oscillators ($\bar{\omega}=1, \Delta=0.25$). No damping.

That a state is marginally stable in a finite volume of parameter space is surprising. In low-dimensional dynamical systems marginally stable states are only found on critical lines separating the stable and the unstable regions, and therefore demands a fine tuning of parameters. Recently, other systems of coupled oscillators showing this kind of marginality have been reported [13,17]. The behavior has been attributed to the generalized time-reversal symmetry of the underlying equations. In Ref. [18] the authors succeeded in giving a rigorous proof of the marginal stability of the stationary state of an *infinite* system of phase rotators. They further showed that in the presence of noise the stationary state becomes linearly stable. For any finite number of oscillators there are inevitable fluctuations in the mean field. The effect of this on the stability properties, however, has not been resolved. Because of the firings the present system is much more intricate. So far we have been unable to find a way to decide whether the "flicker" noise from the firings of individual oscillators in a finite system destroys the marginal stability, although we suspect that it does.

B. The synchronized state

The totally synchronized (in phase) state, where all the oscillators run in unison and therefore has no interaction, is of course also a solution to the system equations. In the following we shall show that this state is unstable for all values of K.

We consider perturbations that divide the population into two families of N_1 and N_2 oscillators, respectively, where the family N_1 is supposed to fire an infinitesimal amount of time before the family N_2. From the equations of motion for the two families we find that the distance δ between their trajectories develops with time according to the equation $\dot{\delta}=-K\delta$ between two firing events. Consequently from Eq. (6) the Lyapunov number taken over a period becomes

$$\lambda=\frac{\omega+\alpha K}{\omega+\alpha K-K}\exp\left[-\frac{K}{\omega}\right] , \tag{16}$$

where we have used the definition $\alpha=N_2/N$.

The stability limit is given by $|\lambda|=1$. This defines two critical values $\alpha_{\pm 1}$ given by

$$\alpha_{\pm 1}=\frac{1}{1\mp\exp(-K/\omega)}-\frac{\omega}{K} . \tag{17}$$

Note that $\lambda=1$ for all values of α if $K=0$. A third critical value α_s is determined by $\omega+\alpha_s K-K=0$ $(K>\omega)$. When this threshold is crossed the slope of the family N_2 becomes negative after the firing of the family N_1. This denotes a superunstable situation. Figure 4 shows for $\omega=1$ the dependence on K of $\alpha_{\pm 1}$ and α_s. Because of the

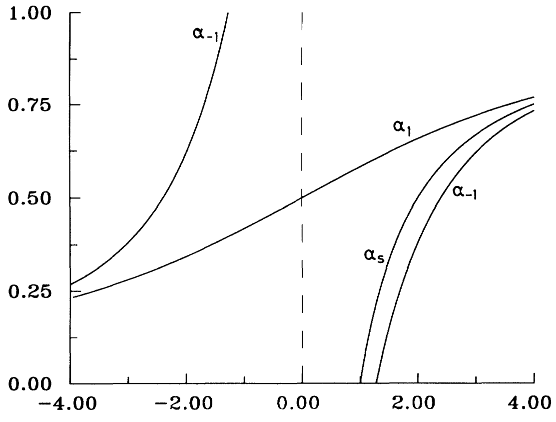

FIG. 4. The critical values $\alpha_{\pm 1}$ and α_s as a function of K for identical oscillators without damping. In the case of damping the critical values move up.

fluctuations there will always be some finite spread in firing times. When a firing sequence starts, the trajectories of the two families will converge as long as α is in the regime above α_1 and for negative values of K below α_{-1}. However, the value of α always has to pass through a region where $|\lambda| > 1$. Here the trajectories will diverge, especially if $K > \omega$ and the critical value α_s is passed. Consequently there will always be a spreading action and this process seems to go on until a uniform distribution of firing times is obtained. This picture has been confirmed on numerical simulations on pools of up to 4097 oscillators. The state shown in Fig. 2 is indeed the steady-state solution, and no self-synchronization is present in the coupled system. Let us remark here that this is the case even for negative values of the coupling constant K.

However, this is a unique consequence of the linear coupling. As soon as a nonlinearity is introduced in the coupling a self-synchronized state jumps into existence for small K. A perturbative approach would thus not suffice for proving the existence of the synchronized state. Depending on the details of the nonlinearity we can in fact turn a repulsive interaction into an attractive one by this mechanism reminiscent of what happens in superconductivity. More details are presented in Appendix B.

C. Including damping

In this section we shall treat the case of identical oscillators with damping included. Under the assumption of s constant the calculation for s can again be performed as a time integral over one period of a single oscillator giving

$$s = \langle x_i \rangle = \frac{\omega + Ks}{K+\Gamma} - \frac{1}{T(K+\Gamma)} , \tag{18}$$

leading to

$$\frac{1}{T} = \omega - \Gamma s , \tag{19}$$

again in agreement with Eq. (10). Inserting this in the expression for the period T [Eq. (12)] finally determines s through an implicit relation:

$$\exp\left[-\frac{K+\Gamma}{\omega - \Gamma s} \right] = 1 - \frac{K+\Gamma}{\omega + Ks} . \tag{20}$$

Since s obviously depends on K, one effect of the damping is therefore to make the period T depend on K. In Fig. 5 we have plotted s and $1/T$ vs K for the parameter values $\omega = 1$ and $\Gamma = 0.5$. The asymptotic value of s is again unity independent of the damping while $\lim_{K\to\infty} 1/T = \omega - \Gamma$.

As before, the state with all oscillators running in unison is a solution to the system equation. The stability of the state can again be investigated by dividing the pool of oscillators into two families separated by an infinitesimal distance. From Fig. 1 we find (in parallel with the treatment without damping) for the Lyapunov number per period

$$\lambda = \frac{\omega + K\alpha}{\omega - \Gamma + K\alpha - K} \exp[-(K+\Gamma)T] . \tag{21}$$

With the oscillators synchronized, the coupling term disappears, and the common period is that for a free single oscillator given by

$$\exp(-\Gamma T) = 1 - \frac{\Gamma}{\omega} . \tag{22}$$

Here, too, we find that $|\lambda| = 1$ defines two critical values $\alpha_{\pm 1}$:

$$\alpha_{\pm 1} = \frac{K+\Gamma}{K[1 \mp (1-\Gamma/\omega)\exp(-KT)]} - \frac{\omega}{K} , \tag{23}$$

where we have used the relation for the period. For a third critical value $\alpha_s = 1 + \Gamma/K - \omega/K$ the state becomes superunstable. As seen, the effect of the damping is to increase the critical values of α, thus rendering the synchronized state even more unstable. This is so as long as the damping itself does not eliminate the oscillations. That the state with s constant is indeed the stable state is again confirmed through numerical simulations.

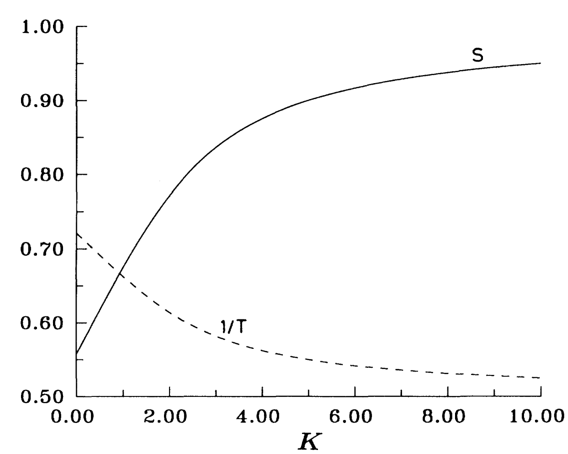

FIG. 5. The mean-field oscillator strength s and the frequency $1/T$ vs the coupling constant K for identical oscillators in the case of damping. The parameters are $\omega = 1, \Gamma = 0.5$.

V. NONIDENTICAL OSCILLATORS, NO MODULATION

Already in the simple case without damping a stability analysis becomes impossible. We shall therefore assume that the stable state is the state with s constant. This assumption is supported by numerical simulations. Many different distributions of natural frequencies are of course possible. We shall show in the following that even the simple form of uniform distribution on a closed interval for some subset of oscillators gives rise to oscillator quiescence, i.e., the firings stop and the state variable becomes constant for a quiescent oscillator. The critical value of the coupling constant K for this to happen is derived.

The equation of motion is now

$$\dot{x}_i=\omega_i-(K+\Gamma)x_i+Ks\ ,\quad \omega_i\in[\bar{\omega}-\Delta,\bar{\omega}+\Delta]\ , \tag{24}$$

with s assumed to be time independent. From this we can deduce that the spread in individual periods is increased by the coupling since the same amount Ks is added to all the natural frequencies but the damping term introduced by the coupling has more time to work on the slower oscillators. However, the average rotation number is in the case of zero damping independent of the coupling strength K, being equal to that of a pool of identical oscillators with frequency $\bar{\omega}$. This is clear from Eq. (10) setting $\Gamma=0$.

Thus quite obviously the coupling does not enforce self-synchronization upon the oscillators as happens in the corresponding case of the "phase rotator model" for an interacting pool of oscillators [10,15]. From the stability analysis of the case with identical oscillators it is evident that the important difference lies in the firing action.

The fixed point x_i^* for the ith oscillator is derived by assuming $\dot{x}_i=0$, giving $x_i^*=(\omega_i+Ks)/(K+\Gamma)$. If for some reason this value becomes smaller than the threshold $T_{\rm top}=1$ the oscillator will stop firing; the oscillator is said to die. This phenomenon is called oscillator quiescence. The first to quiesce will be the one with the smallest natural frequency leading to the following condition of s:

$$s=1+\frac{\Gamma}{K}-\frac{\bar{\omega}-\Delta}{K}\ . \tag{25}$$

To find a relation between the coupling constant K and the spread in natural frequencies we now have to make a self-consistent calculation of s. However, from the calculation of the average rotation number above [Eq. (10)] we have immediately

$$\begin{aligned}\bar{\omega}-\Gamma s&=\frac{1}{N}\sum_{j=1}^{N}\frac{1}{T_i}\\ &=-\frac{K+\Gamma}{N}\sum_{\omega_i\geq\hat{\omega}}\frac{1}{\ln\left[1-\dfrac{K+\Gamma}{\omega_i+Ks}\right]}\ ,\end{aligned} \tag{26}$$

where $\hat{\omega}$ equals either $\bar{\omega}-\Delta$ or $\Gamma+K(1-s)$, whichever is larger. The former value should be used if no oscillators become quiescent. Here we have used Eq. (12) for T_i together with the fact that the rotation number equals zero for a quiescent oscillator.

Solving this and insertion into Eq. (12) would in principle allow us to find the resulting distribution of individual periods.

For $N\gg1$ the above equation can be cast into integral form. With the substitution $y=\omega+Ks-K-\Gamma$ we find

$$\frac{\bar{\omega}-\Gamma s}{K+\Gamma}=\frac{1}{2\Delta}\int_{\hat{\omega}-\Gamma-K(1-s)}^{\bar{\omega}+\Delta-\Gamma-K(1-s)}\frac{dy}{\ln(y+K+\Gamma)-\ln y}\ , \tag{27}$$

from which equation s can be derived by numerical integration. For $\Gamma=0$ the result is displayed as the dashed curve in Fig. 3 which shows s vs K for $\bar{\omega}=1$ and $\Delta=0.25$. For comparison the fully drawn curve shows the corresponding curve for $\Delta=0$. When the condition Eq. (25) on s for the first oscillator to quiesce is fulfilled, the upper limit of the above integral is 2Δ while the lower is zero. The critical line K vs Δ for the first quiescence to occur is shown in Fig. 6 for center frequency $\bar{\omega}=1$ and $\Gamma=0$. Obviously the critical line goes to zero at $\Delta=1$. In the limit of zero spread in the natural frequencies ($\Delta\rightarrow0$) we find a logarithmic growth in the coupling constant $K\rightarrow\bar{\omega}\ln\Delta$ and as expected $s\rightarrow1$.

Figure 7 shows the results for the distribution of individual rotation numbers, R_i, versus ω_i from a numerical simulation on a pool of 4097 oscillators (parameters $\Delta=0.25, \bar{\omega}=1$, $K=0$, 2.3, and 3.5; no damping present). Under these conditions the first oscillator becomes quiescent at a K value around 2.5 according to Fig. 6. This agrees well with the result shown in Fig. 7 where none have become quiescent for $K=2.3$, while about one-seventh of the oscillators have become quiescent for $K=3.5$. Note that at the same time the frequency of the fastest oscillator has increased by approximately 30%. The distribution found here agrees perfectly with that obtained from the analytical solutions, Eqs. (27) and (12). Furthermore, the average rotation number R is found to be 1 as expected.

In the case of the "phase rotator model" the oscillators

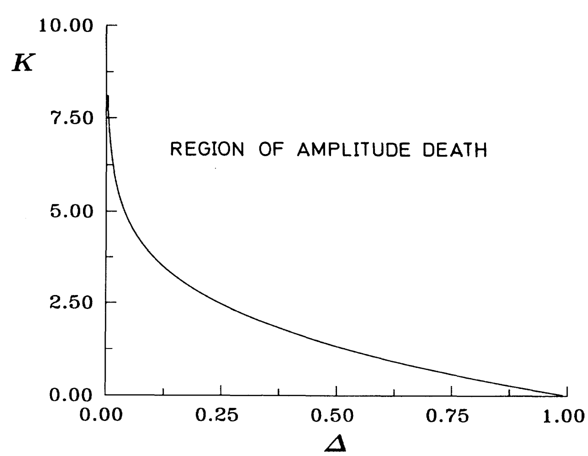

FIG. 6. Critical line in K,Δ space for the first oscillator quiescence to occur. The parameters are $\bar{\omega}=1$, $\Gamma=0$.

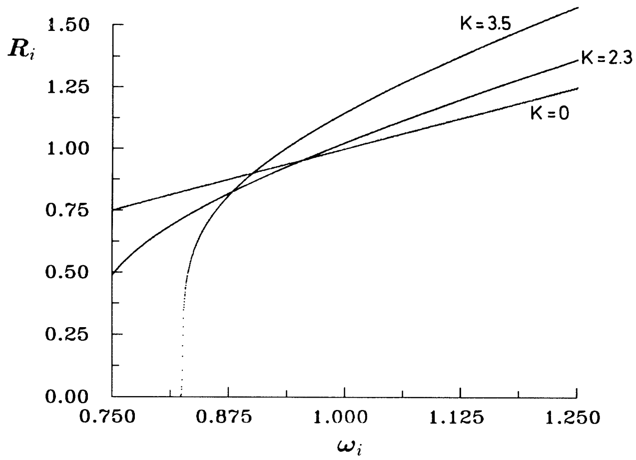

FIG. 7. Rotation number, R_i, for individual oscillators vs ω_i with the coupling constant K as a variable parameter. The parameters are $\bar{\omega}=1, \Delta=0.25, \Gamma=0$.

quiesce collectively. This, of course, can happen here even for zero coupling if the damping is sufficiently high. However, if this were to happen in the present case for $\Gamma=0$, i.e., no damping, Eq. (10) would lead to the trivial case $\bar{\omega}=0$ (and $\Delta=0$). So at minimum one oscillator would survive which using Eq. (26) leads to the condition

$$\bar{\omega}=-\frac{K}{N\ln\left[1-\frac{K}{\bar{\omega}+\Delta+Ks}\right]} . \tag{28}$$

Insertion of the condition for the oscillator with the second highest frequency to quiesce ($K=\bar{\omega}+\Delta+Ks-\Delta/N$) leads to

$$1+\frac{NK}{\Delta}=\exp\left[\frac{K}{N\bar{\omega}}\right] , \tag{29}$$

which besides the trivial solution $K=0$ has a solution for finite K. With $N=4097$, $\Delta=0.25$, and $\bar{\omega}=1$, this gives $K\approx 86\,000$.

VI. IDENTICAL OSCILLATORS WITH MODULATION

We shall proceed by investigating the pool of oscillators under the influence of an external field keeping the interaction linear and attractive in the rest of the paper, i.e., K non-negative. The important question is whether or not phase locking occurs in the system. If phase locking is the case we shall go on to investigate the stability of the phase-locked state (or step as it is often called especially in the literature on Josephson junctions [19]). The first case to be considered is that of modulation on the upper threshold. However, first let us recapitulate some properties for a single oscillator under the influence of an external field (for details see, e.g., [20]).

Consider a single oscillator without damping. Modulation can be applied to either the upper or lower level or both. A special case of the latter is constituted by the modulation on the two levels being in phase and having the same amplitude. In this case the system can be transformed into having the modulation applied to the intrinsic frequency instead.

In the former cases the application of the modulation results in frequency pulling of the oscillator by the external field. If the amplitude is below a certain critical value the resulting state is either phase locked at a rational frequency ratio or quasiperiodic. The critical value occurs when the maximum slope of the modulation becomes identical to the slope of the oscillator state variable at the firing time (without damping this is identical to the intrinsic frequency ω). Above the critical value either the solution itself (modulation on lower level) or the time-reversed solution (modulation on upper level) can have multiple crossings with the modulated level. In (ω,a) space the collection of critical points constitute a critical line of complete phase lock (a complete devil's staircase) having a (complementary) fractal dimension of 0.87. For a sinusoidal modulation the critical line is $\omega=2\pi a$, where a is the amplitude of the modulation. In the case of modulation on the upper level a shadowing effect sets in, forbidding firing in certain time intervals. The result is complete phase lock above the critical line. In the case of modulation on the lower level we get overlapping of the phase-locked regions resulting in hysteresis and chaos above the critical line. In this case an instability will also occur inside the phase-locked regions giving rise to a Feigenbaum bifurcation route to chaos. A stability analysis analogous to that presented above shows that the limits of the phase-locked regions are found by requiring that the Lyapunov number equal 1, while the instability inside the phase-locked region occurs when the Lyapunov number becomes smaller than -1.

If the modulation is on the frequency no phase-locked states exist due to the slopes involved in the calculation of the Lyapunov number being identical before and after the firing event. However, if damping is introduced this symmetry is broken, and phase locking appears. The behavior is parallel to that of the case with modulation on the upper level since the shadowing effect prohibits firing in the time intervals which would introduce chaos into the system. A critical line again occurs at the value of a above which the time-reverse solution can have multiple crossings, i.e., where $\ddot{x}=\dot{x}=0$ for $x=1$. This results in the condition $\omega=\Gamma+(4\pi^2+\Gamma^2)^{1/2}a$.

A. Modulation on the upper level

In what follows we shall ignore the intrinsic damping setting $\Gamma=0$. The equation of motion is then given by Eq. (1) with $\omega_i=\omega$ and $A(t)=0$. Furthermore, we choose a sinusoidal modulation, i.e., $T_{\rm top}=1+a\sin(2\pi t)$, while $T_{\rm bot}=0$. Since all oscillators are assumed to be identical, the obvious solution is the synchronized state. We shall therefore investigate the stability of this state.

For the sake of simplicity we shall assume that for $K=0$, ω is in the range of phase lock on the $1/Q$ step (i.e., $T=Q$). This requires that there exists a time t' such that the following condition is fulfilled:

$$Q\omega=1+a\sin 2\pi t' . \tag{30}$$

From this the limits of the step for $K=0$ are $1-a \le Q\omega \le 1+a$ below the critical line given by $\omega=2\pi a$. As described above, beyond criticality a shadowing effect prohibits firings on part of the upper threshold. Only the lower edge of the step survives while the upper edge is bent inwards (see Ref. [16] for details). For $\omega \le 2\pi a$ we have complete phase lock.

To investigate the stability of the phase-locked solution when the coupling is turned on, we look at the same kind of perturbation as previously dividing the pool of oscillators into two families. From Eq. (6) we find immediately

$$\lambda=\frac{\omega+\alpha K T_{\text{top}}(t')}{\omega+(\alpha K-K)T_{\text{top}}(t')-2\pi a\cos(2\pi t')}\exp(-KQ)$$
$$=\frac{\omega(1+\alpha KQ)}{\omega[1+(\alpha-1)KQ]+2\pi[a^2-(Q\omega-1)^2]^{1/2}}\times\exp(-KQ)\ , \tag{31}$$

where we have used the condition equation (30) for being on the first step.

As in the case without modulation there exists a critical value α_1 ($\lambda=1$) below which the perturbation grows with time, i.e., the locked state becomes unstable, and a critical value α_s where the state becomes superunstable. However, if the amplitude of the modulation is sufficiently large α_1 becomes zero for a finite K and a stable regime exists. Setting $\alpha_1=0$ we find for the critical value a_1 of the amplitude for this to happen:

$$a_1^2=(Q\omega-1)^2+\frac{\omega^2}{4\pi^2}[QK-1+\exp(-QK)]^2\ . \tag{32}$$

For $K=0$, a_1 is seen to coincide with the limits of the step (see above). Likewise we find the critical value a_s by setting $\alpha_s=0$:

$$a_s^2=(Q\omega-1)^2+\frac{\omega^2}{4\pi^2}[QK-1]^2\ , \tag{33}$$

with the further condition $QK>1$. For large values of the coupling constant K the two critical amplitudes converge together. Both are functions of the natural frequency ω. For $\omega=1/Q$, a_1 passes the critical line $\omega=2\pi a$ for QK slightly below 2, while a_s passes for $QK=2$. Displayed in Fig. 8 is the stability region of some of the major $1/Q$ steps (upper curve a_1) and the limit for the superunstable regime (middle curve a_s) versus ω for $K=0.85$. The lines starting at the ω axis show the limits of the stable steps for $K=0$. Also shown is the critical line $\omega=2\pi a$. Thus the step sizes shrink when the coupling is turned on, reminiscent of what happens in the case of globally coupled damped pendulums with random intrinsic pinning under the influence of an external force with modulation [20]. Presumably this means that the critical line for complete phase lock disappears abruptly when the coupling is turned on, since due to the shadow effect steps can only touch but not overlap. However, we have not yet been able to establish the detailed structure. Numerical simulations show that the solution in the unstable regime of the step is incoherent with s modulated by the external frequency. Thus firing times are spread out over all times although their distribution is modulated.

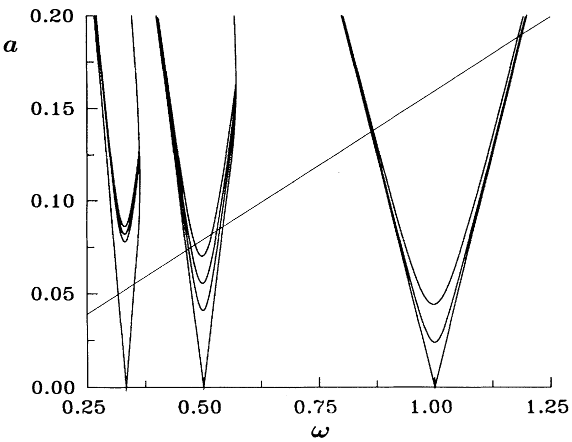

FIG. 8. Critical lines $a_{\pm1}$ and a_s inside some $1/Q$ steps ($Q=1,2,3$) for $K=0.85$. Modulation on the upper level. From above, the order of the critical lines is a_1, a_s, and a_{-1}. Also shown is the critical line $\omega=2\pi a$ for $K=0$. Identical oscillators with $\Gamma=0$.

Let us finally look into whether the inside of the phase-locked region is stable above the critical value a_1. Setting $\lambda=-1$ and $\alpha=0$, results in a condition that is trivially fulfilled for K less than K_{-1} determined by the relation $QK_{-1}=1+\exp(-QK_{-1})$. For larger values of K we find $a\le a_{-1}$ as a condition for the first bifurcation, where a_{-1} is given by

$$a_{-1}^2=(Q\omega-1)^2+\frac{\omega^2}{4\pi^2}[QK-1-\exp(-QK)]^2\ . \tag{34}$$

This critical value is shown as the lower curve in Fig. 8. As seen $a_1>a_s>a_{-1}$ for all values of $K\ge K_{-1}$, even though they converge for $K\to\infty$. The system is therefore stable towards bifurcations for all values of the coupling constant K.

B. Modulation on the lower level

As above, the equation of motion is given by Eq. (1) with $\omega_i=\omega$ and $A(t)=\Gamma=0$. Furthermore, with the modulation on the lower level we have $T_{\text{top}}=1$ and $T_{\text{bot}}=a\sin(2\pi t)$. For the same reasons as above the obvious solution is the synchronized state. We shall therefore investigate the stability of this state.

As before we assume that for $K=0$ we would be phase locked on a $1/Q$ step (i.e., $T=Q$). This requires that there exists a time t' such that the following condition is fulfilled:

$$Q\omega=1-a\sin 2\pi t'\ . \tag{35}$$

Again the limits on the step for $K=0$ are $1-a\le Q\omega\le 1+a$ below the critical line given by $\omega=2\pi a$. As described above, beyond criticality the steps overlap and hysteresis and chaos is found.

The Lyapunov number is given by

$$\lambda = \frac{\omega(1+\alpha KQ) - 2\pi[a^2 - (Q\omega - 1)^2]^{1/2}}{\omega[1+(\alpha-1)KQ]} \exp(-QK) , \tag{36}$$

where we have used the condition equation (35) for being on the $1/Q$ step. From this we immediately observe that the critical value $\alpha_s = (QK - 1)/K$ is independent of a and the system thus is superunstable for $QK > 1$. In this case the modulation is not able to overcome the superinstability present already in the unperturbed system.

Also in this case a critical value α_1 exists ($\lambda = 1$) below which the perturbation grows with time, i.e., the locked state becomes unstable. Again a critical value for the amplitude a_1 can be found where α_1 becomes zero for a finite K and a stable regime exists. Setting $\alpha_1 = 0$ we find for the critical value of the amplitude:

$$a_1^2 = (Q\omega - 1)^2 + \frac{\omega^2}{4\pi^2}[1 + (QK - 1)\exp(QK)]^2 . \tag{37}$$

Likewise we find the critical value a_{-1} by setting $\alpha_{-1} = 0$, where α_{-1} is determined by the condition $\lambda = -1$:

$$a_{-1}^2 = (Q\omega - 1)^2 + \frac{\omega^2}{4\pi^2}[1 - (QK - 1)\exp(QK)]^2 . \tag{38}$$

In this case the instability inside the step occurs for $a \geq a_{-1}$. Crossing this critical line we would therefore expect the synchronized state to bifurcate. As seen, $a_{-1} \geq a_1$ for $QK \leq 1$. For $QK = 1$ (where $\alpha_s = 0$) the two curves become identical. From this we would expect the critical line to be pushed up since no shadow effects exists and steps therefore are allowed to overlap eventually in contrast to the case of modulation on the upper level. Presumably bifurcations and chaos will be present when the critical line is crossed as for a single oscillator. However, at least no $1/Q$ steps will be stable below $Q < 1/K$, and in fact we believe no steps at all will be present in this regime. In Fig. 9 we show the critical lines for some of the major $1/Q$ steps ($K = 0.48$) together with the step sizes with zero coupling in a (ω, a) plot. The conjectures above have been checked by numerical simulations in a few cases and found correct.

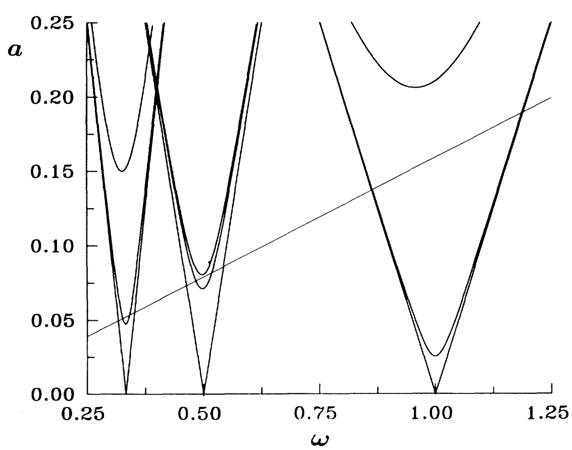

FIG. 9. Critical lines $a_{\pm 1}$ and a_s inside some $1/Q$ steps ($Q = 1,2,3$) for $K = 0.48$. Modulation on the lower level. From above, the order of the critical lines is a_{-1} and a_1 on the $Q = 1$ and 2 steps, while the order is reversed on the $Q = 3$ step. However, this step is in the region $KQ > 1$ and is therefore unstable for all values of a. Also shown is the critical line $\omega = 2\pi a$ for $K = 0$. Identical oscillators with $\Gamma = 0$.

C. Modulation on frequency

As we have seen without modulation, self-synchronization does not take place in the coupled system. Instead the coupling throws the system into a state where firing times are evenly distributed and a damping term is present. For a single isolated oscillator it is known that such a damping term is sufficient for creating phase lock to an external signal modulating the natural frequency. A natural question to ask is therefore whether the damping introduced by the coupling is enough to secure phase lock in the present system. However, since all oscillators are identical, phase lock would force them into running in unison and the damping, and consequently the phase lock, would disappear. Thus the state with damping would reappear. This apparent paradox is cleared up by looking at Eq. (10) for the average rotation number. Without damping (i.e., $\Gamma = 0$) we see from this that the average rotation number is equal to the natural frequency ω *independent* of the modulation. Thus the solution for the coupled system must be one where s is modulated in such a way as to absorb the effect of the external modulation. However, this could still force the system into a state of self-synchronization even though no phase lock takes place.

To investigate this possibility we assume that all oscillators run in unison and that ω is irrational. The state is then quasiperiodic with a uniform distribution of firing times t^* mod 1. In order to examine the stability one therefore has to consider the time-averaged Lyapunov number. The period T is given by $\omega T - (a/2\pi)\cos(2\pi T) + a/2\pi = 1$. As usual the Lyapunov number is found from Eq. (6) giving for the average Lyapunov number per period:

$$\langle \lambda(t) \rangle = \left\langle \frac{\delta_2}{\delta_1} \right\rangle \exp(-KT)$$
$$= \left\langle \frac{\omega + a\sin(2\pi t) + K\alpha}{\omega + a\sin(2\pi t) + K\alpha - K} \right\rangle \exp(-KT) . \tag{39}$$

If $|\langle \lambda \rangle| > 1$ for $\alpha = 0$, the state is always unstable. A necessary and sufficient condition for stability is therefore

$$\left\langle 1 + \frac{K}{\omega + a\sin(2\pi t) - K} \right\rangle \exp(-KT) \leq 1 . \tag{40}$$

The time-averaging integral can be performed and a numerical evaluation shows that the above statement is always false. Thus the synchronized state is always unstable.

If the denominator in Eq. (39) becomes negative the state becomes superunstable. The condition for this to happen is $\alpha = 1 - [\omega + a\sin(2\pi t)]/K$. Since ω is assumed irrational every firing time modulo 1 is visited. The instability therefore sets in for the maximum value possible, i.e., $\alpha_s = 1-(\omega - a)/K$. The critical line $\alpha_s = 0$ is then given by $-a+\omega = K$.

VII. NONIDENTICAL OSCILLATORS WITH MODULATION

We shall now turn to the system of nonidentical oscillators with modulation applied, first treating the case of modulation on the upper level.

A. Modulation on the upper level

The equations of motion are given by Eq. (1) with $A(t)=0$. Furthermore, we shall ignore damping in this section. In the absence of coupling all oscillators can be phase locked to the same step if the spread in frequency is sufficiently small. If the spread is larger the rotation numbers of the individual oscillators will for any given values of a and Δ lie on a devil's staircase which for some oscillators may be incomplete, and for others complete if the critical line is crossed.

The same picture is found in the case of finite coupling, $K \neq 0$, together with the spreading of individual frequencies and oscillator quiescence already observed for nonidentical oscillators in the absence of modulation. In Fig. 10 we show the individual rotation number R_i versus the natural frequency ω_i for $K=0$ from a numerical simulation on a pool of 1025 oscillators with $\Delta = 0.75$, $\varpi = 1$, and $a = 0.13$. A simulation on the same system for $K=2.5$ is displayed in Fig. 11. As seen oscillator quiescence and frequency spreading is observed as expected together with a devil's staircase, even though in fact more oscillators are now phase locked than before the interaction is turned on. A critical point is still found dividing the devil's staircase in a complete and an incomplete section, but the oscillators are pushed across the critical line deeper inside the regime of complete phase lock. However, the question is whether the state of the system as a whole can be phase locked. Deemed from extensive numerical simulations, the answer to this question is affirmative. The coupled system can indeed be in a phase-locked state with a rational average rotation number provided a single step is sufficiently large to accommodate all oscillators. However, the simulations also show that the firing times are spread out so no two oscillators fire simultaneously. The oscillators therefore cannot be said to be synchronized.

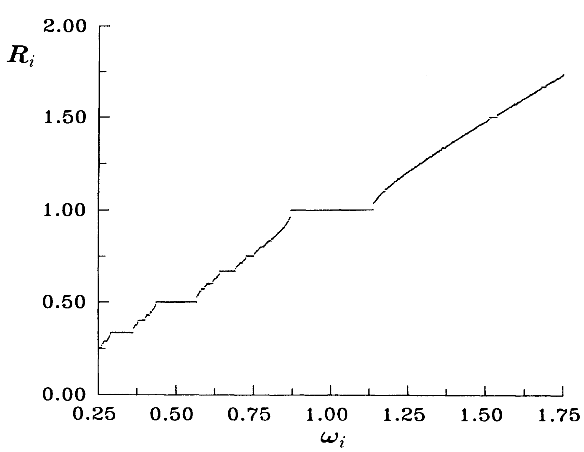

FIG. 10. R_i vs ω_i for $K=0$, modulation on the upper level, no damping. The parameters are $\varpi = 1$, $\Delta = 0.75$, $a = 0.13$.

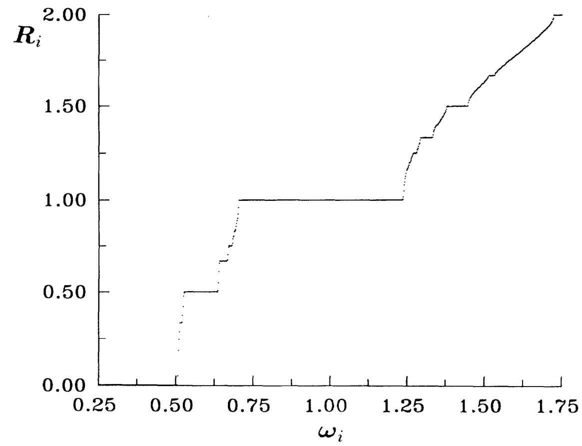

FIG. 11. R_i vs ω_i for $K=2.5$, modulation on the upper level, no damping. The parameters as in preceding figure.

An interesting question is whether there is a lower limit to phase lock for the averaged system even if all oscillators could be accommodated on a single step. However, the lower limit found in the case of identical oscillators may be strictly connected with the degeneracy present there. The existence of a critical line is also connected to this question. If a lower limit exists one would expect the critical line to disappear for the average rotation number, and the step structure for the system as a whole will probably consist of isolated phase-locked islands interspersed with quasiperiodic states. We have not been able to resolve these questions so far.

With respect to modulation on the lower level the situation is much the same as above as long as the critical line $\omega = 2\pi a$ is not crossed by any individual oscillator. This case therefore will not be treated separately. If the critical line is crossed, hysteresis and chaos occur for the individual oscillators. Whether this means a complete breakdown for the possibilities of phase lock for the averaged system is not known.

B. Modulation on frequency

The equation of motion is now

$$\dot{x}_i = \omega_i - (K+\Gamma)x_i + Ks + a\sin(2\pi t) ,$$
$$\omega_i \in [\varpi - \Delta, \varpi + \Delta] . \quad (41)$$

As in the case of identical oscillators we know that if the intrinsic damping is ignored, due to the relation equation (10) for the average rotation number no synchronization to the external modulation will take place for the system as a whole. For $K=0$ the individual oscillators will not phase lock. We shall therefore treat the system without intrinsic damping as the most interesting case.

The system has to some extent been investigated by numerical simulations. A simple result for K somewhat above 1 seems to be that the mean field s oscillates out of phase with the external modulation, i.e., $s(t)=b-s_0\sin(2\pi t)$, where $Ks_0<a$. This result has been checked on a pool of 2049 oscillators for many different combinations of K, Δ, $\bar{\omega}$, and a.

We have run simulations for finite values of the coupling as well as without coupling on this system. As expected there is no sign of phase lock with the coupling turned off. A simulation of 2049 oscillators with $K=3$, $\bar{\omega}=1$, $\Delta=0.3$, and $a=0.5$ is displayed in Fig. 12. From the results for $a=0$ in Sec. V we would expect a spread in individual rotation numbers and oscillator quiescence to occur. This does happen. However, most surprisingly the individual oscillators seem to phase lock to the external modulation even though we know that the average rotation number is independent of the modulation. Furthermore, a devil's staircase seems to exist as when the modulation is on the levels. To investigate this phenomenon further, we shall look at the Lyapunov number for a single oscillator.

Following the usual analysis we find for the Lyapunov exponent related to the Lyapunov number by $\lambda=\exp(\Lambda)$:

$$\Lambda_i=-K+\lim_{t\to\infty}\frac{1}{t-t_0}\times\sum_{t_j}\ln\left|\frac{\omega_i+Ks(t_j)+a\sin(2\pi t_j)}{\omega_i-K+Ks(t_j)+a\sin(2\pi t_j)}\right|, \tag{42}$$

where t_j is the time for the jth firing. In Fig. 13 we show the numerical result for the Lyapunov exponent corresponding to Fig. 12 versus the natural frequency. As seen, there is a regime where the individual oscillators are completely phase locked below the first step and a regime where the individual oscillators are in either phase-locked or quasiperiodic states starting at $\omega_i\approx1.08$. Thus we find a devil's staircase with a critical point for the distribution of individual rotation numbers. If we look at the behavior of the Lyapunov exponent for a collection of oscillators phase locked to the same step we observe that no two oscillators have the same exponent. The meaning of this becomes more clear if we plot the distribution of first firing times τ_i after a given time t' versus natural frequency as shown in Fig. 14 corresponding to the simulation shown in Fig. 12. Again the distribution is such that no two oscillators phase locked on the same step fire simultaneously. Thus even though the oscillators having the same rotation number are all phase locked to the modulation, they cannot be said to be locked to each other. There is, of course, an ambiguity built into this construction since for instance on a P/Q step there are Q different periods of the external force that a single oscillator phase locked on such a step can fire on. Thus the first firing times on such a step will fall in Q distinct groups. For an irrational (quasiperiodic orbit) rotation number a continuous range of firing times will be observed.

The critical point that separates the completely phase-

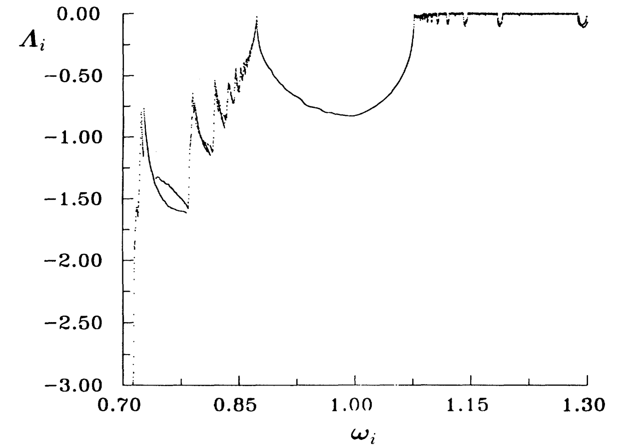

FIG. 13. Lyapunov exponent referring to Fig. 12. The ambiguity found inside the phase-locked steps is due to the finite values of the time steps and number of oscillators.

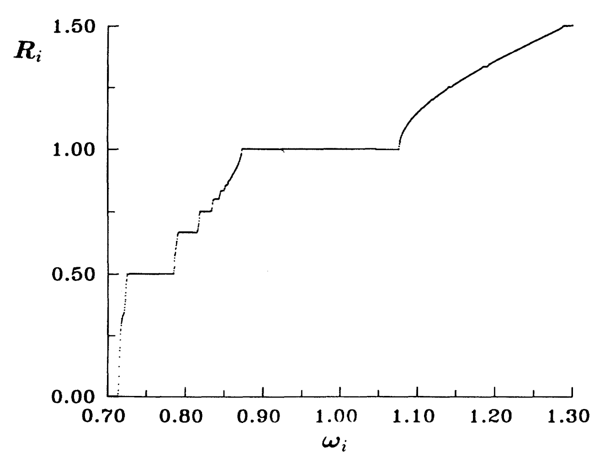

FIG. 12. R_i vs ω_i for $K=3$, modulation on the current, no damping. The parameters are $\bar{\omega}=1$, $\Delta=0.3$, $a=0.5$.

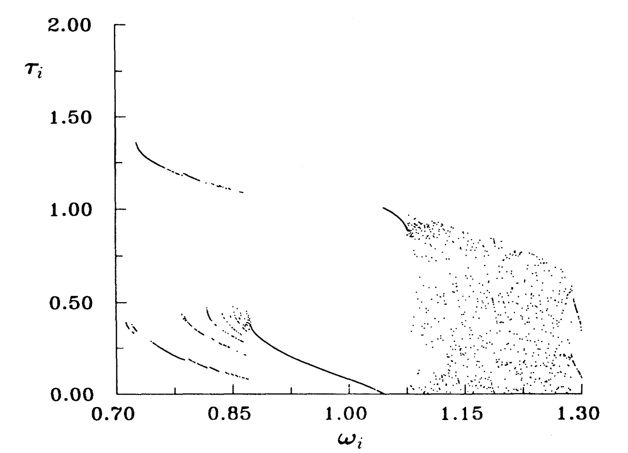

FIG. 14. Distribution of first firing times (parameters as in Fig. 12).

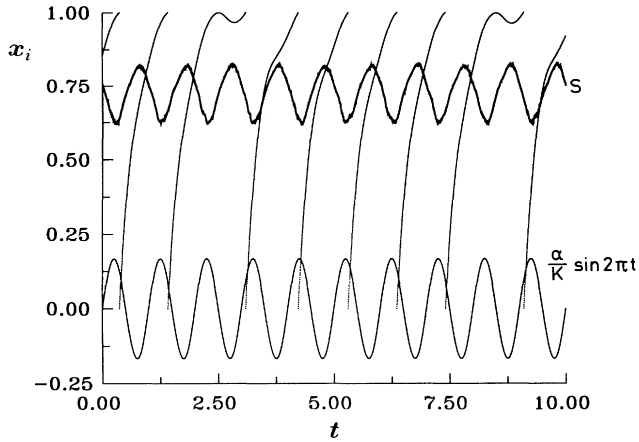

FIG. 15. Time development of the slowest oscillator and of the mean-field oscillator strength s (parameters as in Fig. 12). The lowest curve shows the external field for ease of comparison.

locked regime from the regime with quasiperiodic and phase-locked states can be determined from the observation that the mean field s is out of phase with the modulation. The condition that specifies which oscillator is at the critical point is $\dot{x}_i=\dot{x}_i=0$ for $x_i=1$. Inserting the above-mentioned equation for s in the equation for x_i and taking the minimum we find

$$\omega_i^c = K + a - K(b+s_0) = K + a - Ks_{\max} , \tag{43}$$

for the critical natural frequency. The value of $s_{\max}$ can be taken to be approximately 0.81 from Fig. 15, where we have shown the time development for the slowest oscillator and the mean field s corresponding to the simulation presented in Fig. 12. The resulting value for the critical natural frequency corresponds well with that obtained from the calculation of the Lyapunov exponent, showing that the mechanism at work is indeed the shadowing effect as for a single isolated oscillator [16].

VIII. CONCLUSION

We have treated a large system of globally and homogeneously connected relaxation oscillators, where the coupling is assumed linear and continuous. This allows for a mean-field approach, which is the main reason for considering a linear coupling. One very important question when discussing large pools of interacting oscillators is whether self-synchronization take place in the system. Under the specified circumstances the answer is shown to be negative. However, we show that for identical oscillators this is entirely due to the linear form of the coupling. If the coupling is nonlinear, under the right circumstances even a repulsive interaction can be transformed into an effective attractive interaction, due to the instantaneous resetting. We find that synchronization can also take place for nonidentical oscillators but more work needs to be done on this problem.

The state found to be the preferred state is the stationary or incoherent state having an even distribution of firing times. The stability of this state is difficult to study whether by analytical means or through numerical simulations. However, simulations with small time steps and with initial conditions close to the stationary state show only a slow decay of perturbations. Furthermore, the simulations show some evidence for the time scale of the decay to increase with the number of oscillators. Analytically the system is hard to attack partly due to the discontinuous firings partly due to the high degeneracy encountered when the oscillators are identical. However, our investigation leads us to the conclusion that the stationary state is marginally stable.

For nonidentical oscillators the introduction of the coupling results in amplitude quiescence. However, in contrast to the case of limit cycle oscillators no collective amplitude quiescence takes place, but instead an increase in the distribution of individual frequencies. We have determined the critical line for the first oscillator to quiesce. Also in this case the stationary state seems to be the preferred state.

The system has furthermore been investigated in the presence of an external field. One important result for identical oscillators is that the phase-locked regions for the average system shrinks and one has to go to finite values of the amplitude of the external field before phase locking is achieved. Below this level our investigation indicates that the stationary state as usual is the preferred state. For modulation on the upper level our investigations indicate that the critical line encountered for a single oscillator for transition to complete phase lock disappears abruptly, when the field is turned on. For modulation on the lower level, the critical line for transition to hysteresis and chaos is pushed to higher values of the field amplitude but still seems to exist. More work is needed to give clear answers concerning this.

When the oscillators are nonidentical we find that for modulation on the upper level the entire system can be phase locked provided that all individual oscillators can be accommodated on the same step. If not, the rotation numbers of the individual oscillators constitute a devil's staircase, part of which can be complete, part of which can be incomplete if the critical line is crossed. A surprising result is that even when the entire system is phase locked to the external field no two oscillators fire at the same time. The individual oscillators thus in a sense cannot be said to be locked together. Another surprise is encountered for modulation on the frequency. Just as for the single oscillator the entire system is unaffected by the modulation. However, the individual oscillators are due to the coupling forced into a state where their individual rotation numbers constitute a devil's staircase. Again no two oscillators fire at the same time.

APPENDIX A: NUMERICAL METHODS

For identical oscillators the firing events cause a problem for the numerical integration. In general, a disturbance will converge between firings and discontinuously diverge at the firings. If the firings simply are implemented by resetting x when the upper level is exceeded oscillators firing in the same time step will be synchronized forever. This nucleation creates groups of oscillators that eventually may merge into the fully synchronized state. Even when an interpolation scheme is used the contribution from the firings to the stability exponent will always by underestimated in the simulation. When studying the stability of the stationary state the problem is serious. If this is marginally stable, as we claim, it will always appear unstable in the simulations. Choosing the time step small enough allowing for many time steps between firings, the state will appear metastable. A given perturbation will decay and the stationary state exists for some time before breaking up into clusters. It is worth noting the resemblance between pulse coupling and the spurious consequence of a finite time step. Arguments analogous to those above suggest that when both pulse coupling and continuous coupling is present, the long-time behavior is controlled by the pulse coupling, and the stable state is fully synchronized.

The numerical integration has in most cases been performed with an Euler algorithm. The size of the pool was from 129 to 4097 oscillators. The large pools were mainly used to get good resolution of the step structures. Time steps were chosen between 1/128 and 1/4096 with the large steps sufficient when the oscillators had distributed frequencies. We note that for a given time step the time for a simulation increases linearly with the number of oscillators N when using Eq. (3) compared to N^2 when using Eq. (1).

A few of the simulations were checked with a fourth-order Runge-Kutta algorithm. No significant discrepancy was noted compared to the Euler algorithm.

APPENDIX B: NONLINEAR COUPLING

In this Appendix we shall briefly treat the case of introducing a nonlinearity in the coupling. We shall only consider the simplest case of identical oscillators without damping and without modulation. The equation of motion can then be written as

$$\dot{x}_i=\omega+\frac{K}{N}\sum_{j=1}^{N} g(x_j-x_i) \ , \tag{B1}$$

where the coupling function $g(x)$ is assumed continuous and odd, i.e., $g(x)=-g(-x)$ and $g(0)=0$. Furthermore, $g(1)=1$ and $g'(0)>0$, so that the interaction is attractive if $K>0$.

We shall demonstrate in the following that the introduction of a nonlinearity into the coupling abruptly brings about self-synchronization. To calculate the Lyapunov number for the synchronized state we as usual divide the pool into two families. Referring to Eq. (6) we first have to calculate the slopes h_1 and h_2:

$$\begin{aligned} h_1&=\omega+\frac{KN_1}{N}g(-1)=\omega-K(1-\alpha) \ , \\ h_2&=\omega+\frac{KN_2}{N}g(1)=\omega+K\alpha \ . \end{aligned} \tag{B2}$$

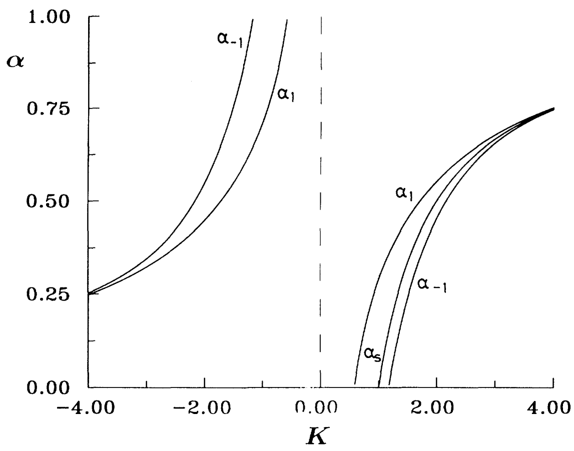

FIG. 16. The critical values $\alpha_{\pm1}$ and α_s as function of K for identical oscillators ($\omega=1$) without damping but with a nonlinear coupling with $g'(0)=1.5$. A synchronized state appears for positive K values smaller than approximately 0.6.

We also need to calculate the change in the perturbation in between firings. Using that $x_j-x_i \ll 1$ everywhere in between firing events we find for the time development of the perturbation:

$$\begin{aligned} \dot{x}_1-\dot{x}_2&=\frac{K}{N}\sum_{j=1}^{N}[g(x_j-x_1)-g(x_j-x_2)] \\ &\approx -Kg'(0)(x_1-x_2) \ . \end{aligned} \tag{B3}$$

Combining these results the Lyapunov number becomes

$$\begin{aligned} \lambda&=\frac{h_2}{h_1}\exp[-KTg'(0)] \\ &=\frac{\omega+K\alpha}{\omega-K(1-\alpha)}\exp\left[-\frac{Kg'(0)}{\omega}\right] \ . \end{aligned} \tag{B4}$$

The stability criterion $|\lambda|\le 1$ defines as usual two critical

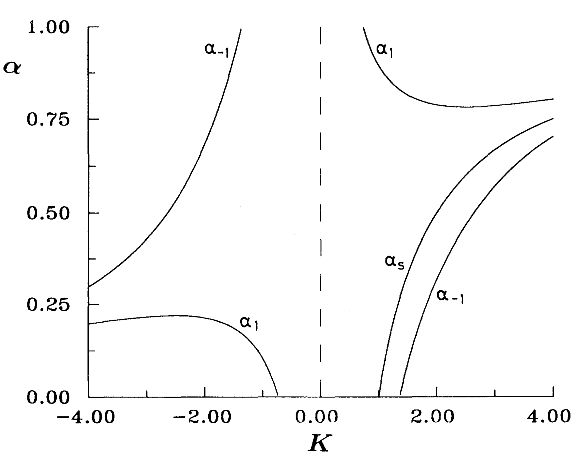

FIG. 17. The critical values $\alpha_{\pm1}$ and α_s as function of K for identical oscillators ($\omega=1$) without damping but with a nonlinear coupling with $g'(0)=0.75$. A synchronized state appears for negative K values greater than approximately -0.75.

values $\alpha_{\pm 1}$ given by

$$\alpha_{\pm 1} = \frac{1}{1 \mp \exp\left[-\frac{K}{\omega} g'(0)\right]} - \frac{\omega}{K} , \tag{B5}$$

which evidently depends on the nature of the nonlinearity. The critical value α_s for the superinstability is given by $\alpha_s = 1 - \omega/K$. In Figs. 16 and 17 we have plotted the critical values as function of K for $g'(0) > 1$ and $g'(0) < 1$, respectively.

If $\alpha_1 = 0$ for a finite value of K the self-synchronized state will be stable. This leads to the following relation:

$$1 - \frac{K}{\omega} = \exp\left[-\frac{Kg'(0)}{\omega}\right] . \tag{B6}$$

If $g'(0) > 1$, this equation has a solution K_c in the interval $]0,\omega[$ so that the synchronized state is stable for every K in the interval $]0,K_c[$. Even more surprising, if $g'(0) < 1$, then the equation has a solution K_c in $]-\infty,0[$ so that the synchronized state is stable for every K in $]K_c,0[$. Thus the introduction of the nonlinearity abruptly brings about self-synchronization.

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